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NUMERICAL MODELING OF STRESS STATES IN TWO-DIMENSIONAL PROBLEMS OF ELASTICITY BY THE LAYERS METHOD

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We are examining the iterative solution of problems of elasticity by using a sandwich of layers to represent the region in which we seek the solution. We will use the layer equations in [1-3], the order of these equations being independent of the conditions for the displacements and stresses on the surfaces of the layers. The number of layers is successively doubled during the iteration. In the first approximation, the region is represented in the form of a single layer.

As examples, we will model the stress state near a slit and edge effects in the stress state of elastic layers between rigid slabs.

1. Equations of a Layer. We use the equations of a layer in the first approximation [1] below. In their derivation, the equations of the two-dimensional problem of the theory of elasticity

$$\begin{gathered}
\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \ \sigma_{12} = a \frac{\partial u_1}{\partial x_1} + b \frac{\partial u_2}{\partial x_2}, \\
\sigma_{22} = a \frac{\partial u_2}{\partial x_2} + b \frac{\partial u_1}{\partial x_1}, \ \sigma_{12} = \mu \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right), \\
x_1 \in [0, L], \ x_2 \in \left[x_2^0 - \frac{h_2}{2}, x_2^0 + \frac{h_2}{2} \right], \\
a_i^- \sigma_{i1} + b_i^- u_i = r_i^-, \ x_1 = 0, \ a_i^+ \sigma_{i1} + b_i^+ u_i = r_i^+, \ x_1 = L, \\
c_i^\pm \sigma_{i2} + d_i^\pm u_i = g_i^\pm, \ x_2 = x_2^0 \pm \frac{h_2}{2}, \ i, j = 1, 2
\end{gathered} \tag{1.1}$$

are approximated by the equations [1, 2]

$$\begin{gathered}
\frac{\partial \sigma_{ij}'}{\partial x_j} + f_i' = 0, \ \sigma'_{11} = \sum_{k=0}^{1} \overset{(k)}{\sigma}_{11} P_k, \ \sigma'_{12} = \sum_{k=0}^{2} \overset{(k)}{\sigma}_{12} P_k, \\
\sigma'_{21} = \overset{(0)}{\sigma}_{12}, \ \sigma'_{22} = \sum_{k=0}^{1} \overset{(k)}{\sigma}_{22} P_k, \ \overset{(k)}{\sigma}_{ij} = \frac{1}{2}(1 + 2k) \int_{-1}^{1} \sigma_{ij} P_k d\zeta, \\
\zeta = \frac{2}{h_2}(x_2 - x_2^0), \ \sigma_{11} = a \frac{\partial u'_1}{\partial x_1} + b \frac{\partial u''_2}{\partial x_2}, \\
\sigma_{22} = a \frac{\partial u''_2}{\partial x_2} + b \frac{\partial u'_1}{\partial x_1}, \ \sigma_{12} = \mu \left(\frac{\partial u''_1}{\partial x_2} + \frac{\partial u'_2}{\partial x_1} \right), \\
u'_1 = \sum_{k=0}^{1} \overset{(k)}{u}_1 P_k, \ u''_1 = \sum_{k=0}^{3} \overset{(k)}{u}_1 P_k, \ u'_2 = \overset{(0)}{u}_2, \ u''_2 = \sum_{k=0}^{2} \overset{(k)}{u}_2 P_k, \\
f'_1 = \sum_{k=0}^{1} (f_1)^{(k)} P_k, \ f'_2 = (f_2)^{(0)};
\end{gathered} \tag{1.2}$$

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$$c_i^{\pm}\sigma'_{i2} + d_i^{\pm}u''_i = g_i^{\pm},\ x_2 = x_2^0 \pm \frac{h_2}{2};\tag{1.3}$$

$$a_i^-\sigma'_{i1} + b_i^-u'_i = r'^-_i,\ x_1 = 0,\ a_i^+\sigma'_{i1} + b_i^+u'_i = r'^+_i,\ x_1 = L,$$

$$r'^-_1 = \sum_{k=0}^{1} (r_1^-)^{(k)}P_k,\ r'^+_1 = \sum_{k=0}^{1} (r_1^+)^{(k)}P_k,\ r_2'^- = (r_2^-)^{(0)},\ r'^+_2 = (r_2^+)^{(0)}.\tag{1.4}$$

Here, L and h_2 are the length and thickness of the layer; $x = x_2^0$ is the middle surface of the layer; $a_i^{\pm}$ and $b_i^{\pm}$ are constants satisfying the conditions $a_i^+b_i^+ \geq 0$ and $a_i^-b_i^- \leq 0$; $c_i^{\pm}$ and $d_i^{\pm}$ are piecewise-constant functions equal to zero or unity; $r_i^{\pm}(x_2)$, $g_i(x_1)$, and $f_i(x_1, x_2)$ are assigned functions; P_k represents Legendre polynomials in ζ; k is the degree of each polynomial; the symbol $(\ )^k$ denotes the k-th coefficient of the Legendre-polynomial expansion. In plane-strain problems

$$a = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)},\quad b = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

while in problems concerning a plane stress state

$$a = \frac{E}{1-\nu^2},\quad b = \frac{\nu E}{1-\nu^2}$$

(E is the elastic modulus and ν is the Poisson's ratio).

The functions $\sigma_{ij}^{(L)}$ and $u_i^{(k)}$ in (1.2)-(1.4), whose derivatives with respect to x_1 are contained in (1.2), will be conditionally referred to as the main functions. All other $\sigma_{ij}^{(L)}$ and $u_i^{(k)}$ will be called auxiliary functions. The main functions are

$$\overset{(0)}{\sigma}_1,\ \overset{(0)}{\sigma}_{21},\ \overset{(1)}{\sigma}_{11},\ \overset{(0)}{u}_1,\ \overset{(0)}{u}_2,\ \overset{(1)}{u}_1,\tag{1.5}$$

where $h_2\sigma_{11}^{(0)}$, $h_2\sigma_{21}^{(0)}$, and $(2/3)h_2^2\sigma_{11}^{(1)}$ are the forces and bending moment in the sections of the layer; $u_1^{(0)}$, $u_2^{(0)}$, and $u_1^{(1)}/h_2$ are the displacements averaged over the thickness of the layer and the angle of rotation of the sections.

The solution of Eqs. (1.2)-(1.3) reduces to the integration of a system of sixth-order differential equations for the main functions. The gėneral solution of this system was presented in [3] for different types of conditions on the surfaces of the layer. Use of the general solution together with boundary conditions (1.4) makes it possible to obtain the solution to a wide range of problems – including contact problems [2, 3] – that satisfy all of the continuity conditions for the forces, moments, displacements, and angles of rotation at the boundaries of the sections, with different conditions for the stresses and displacements on the surfaces of the layer as a whole.

The system of differential equations in the main functions (1.5), being of the same order regardless of the conditions for the displacements and the stresses on the surfaces of the layer, can be obtained by means other than Eqs. (1.2). For example, the system can be obtained by the methods described in [4].

2. Stiffness Matrix of an Element of the Layer. Let us examine an element of the layer $-1 \leq \xi \leq 1$, $\xi = (2/h_1)(x_1 - x_1^0)$, where h_1 is the length of the element and x_1^0 is the coordinate of its middle. We designate

$$\mathbf{T}_1 = \begin{bmatrix} h_2\overset{(0)}{\sigma}_{11} \\ h_2\overset{(0)}{\sigma}_{21} \\ \frac{2}{3}h_2^2\overset{(1)}{\sigma}_{11} \end{bmatrix}_{\xi=1},\ \mathbf{T}_2 = \begin{bmatrix} h_1\sigma'_{22} \\ h_1\sigma'_{12} \end{bmatrix}_{\zeta=1},-\mathbf{T}_3 = \begin{bmatrix} h_2\overset{(0)}{\sigma}_{11} \\ h_2\overset{(0)}{\sigma}_{21} \\ \frac{2}{3}h_2^2\overset{(1)}{\sigma}_{11} \end{bmatrix}_{\xi=-1},-\mathbf{T}_4 = \begin{bmatrix} h_1\sigma'_{22} \\ h_1\sigma'_{12} \end{bmatrix}_{\zeta=-1};\tag{2.1}$$

$$\mathbf{U}_1 = \begin{bmatrix} \overset{(0)}{u}_1 \\ \overset{(0)}{u}_2 \\ \overset{(1)}{u}_1{}^2/h_2 \end{bmatrix}_{\xi=1},\ \mathbf{U}_2 = \begin{bmatrix} u''_1 \\ u''_2 \end{bmatrix}_{\zeta=1},\ \mathbf{U}_3 = \begin{bmatrix} \overset{(0)}{u}_1 \\ \overset{(0)}{u}_2 \\ \overset{(1)}{u}_1{}^2/h_2 \end{bmatrix}_{\xi=-1},\mathbf{U}_4 = \begin{bmatrix} u''_1 \\ u''_2 \end{bmatrix}_{\zeta=-1}.\tag{2.2}$$

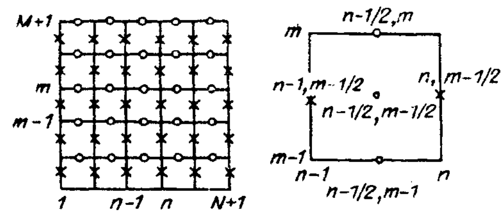

Fig. 1

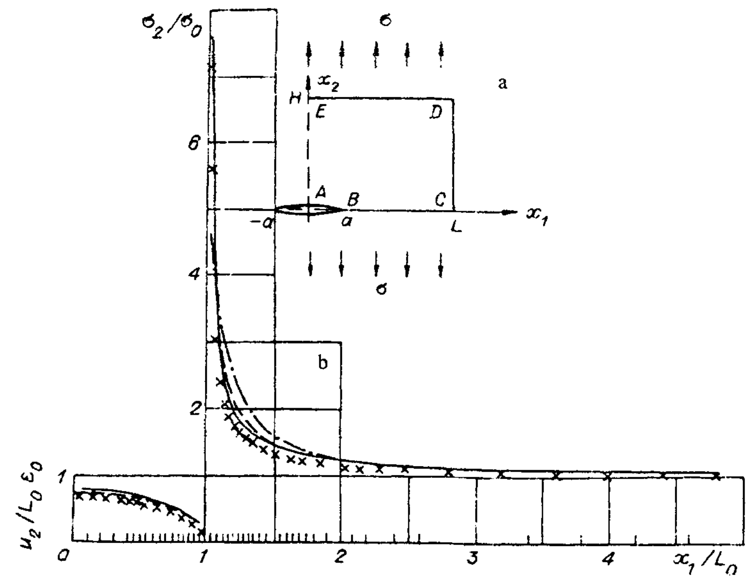

Fig. 2

Replacing the main functions in (1.2) by the half-sum of their values for $\xi = -1$ and 1 and replacing the derivatives with respect to x_1 by the difference of these values divided by h_1, we obtain a system of algebraic equations in the quantities (2.1), (2.2) if we assume that the auxiliary functions are constant within the element. This system can be solved for T_i:

$$T_i = A_{ik}U_k + F_i,\ i = 1, 2, 3, 4.$$

The matrices A_{ik} form the symmetric positive-definite stiffness matrix K of the element:

$$[T_1, T_2, T_3, T_4]^T = [K][U_1, U_2, U_3, U_4]^T + F. \tag{2.3}$$

Another method of obtaining Eqs. (2.3) was presented in [5].

3. Algebraic Equations of the Sandwich. Let the region in which we are seeking the solution to the two-dimensional elastic problem be a rectangle. We will examine it as a sandwich of M layers, each of which consists of N elements (Fig. 1a). Figure 1b shows the numeration of the elements and their sides. The quantities in (2.1)-(2.2) that pertain to the element $(n - 1/2, m - 1/2)$ will be designated by the subscript $n-1/2$ and superscript $m-1/2$. We also introduce the notation:

$$U_1^{m-1/2} = (U_3)_{1/2}^{m-1/2},\ U_n^{m-1/2} = (U_1)_{n-1/2}^{m-1/2} = (U_3)_{n+1/2}^{m-1/2},\ U_{N+1}^{m-1/2} = (U_1)_{N+1}^{m-1/2},$$
$$n = 2,\dots,N,\ m = 2,\dots,M+1,$$
$$U_{n-1/2}^{1} = (U_4)_{n-1/2}^{1/2},\ U_{n-1/2}^{m} = (U_2)_{n-1/2}^{m-1/2} = (U_4)_{n-1/2}^{m+1/2},\ U_{n-1/2}^{M+1/2} = (U_2)_{n-1/2}^{M+1/2},$$
$$n = 2,\dots,N+1,\ m = 2,\dots,M.$$

The conditions of continuity of the forces and moments at the interfaces of adjacent elements

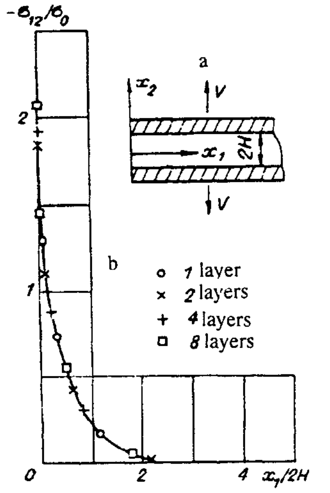

Fig. 3

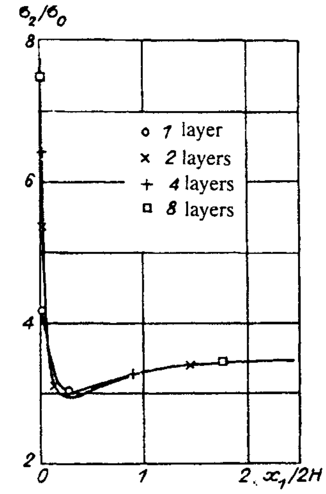

Fig. 4

$$(T_1)^{m-1/2}_{n-1/2} + (T_3)^{m-1/2}_{n+1/2} = 0,\ n = 2,\ldots,N,\ m = 2,\ldots,M+1; \tag{3.1}$$

$$(T_2)^{m-1/2}_{n-1/2} + (T_4)^{m+1/2}_{n-1/2} = 0,\ n = 2,\ldots,N+1,\ m = 2,\ldots,M, \tag{3.2}$$

conditions that follow from (1.3),

$$\begin{gathered} C^{+}_{n-1/2}(T_2)^{M+1/2}_{n-1/2} + D^{+}_{n-1/2}U^{M+1/2}_{n-1/2} = G^{M+1/2}_{n-1/2}, \\ C^{-}_{n-1/2}(T_4)^{1/2}_{n-1/2} + D^{-}_{n-1/2}U^{1/2}_{n-1/2} = G^{1/2}_{n-1/2},\ n = 2,\ldots,N+1 \end{gathered} \tag{3.3}$$

and conditions that flow from (1.4),

$$\begin{gathered} C_0^{m-1/2}(T_3)^{m-1/2}_{1/2} + D_0^{m-1/2}U^{m-1/2}_{1/2} = G_0^{m-1/2}, \\ C_L^{m-1/2}(T_1)^{m-1/2}_{N+1/2} + D_L^{m-1/2}U^{m-1/2}_{N+1/2} = G_L^{m-1/2},\ m = 2,\ldots,M+1 \end{gathered} \tag{3.4}$$

form a system of algebraic equations in $U_n^{m-1/2}$, $U^m_{n-1/2}$ – the vectors of the displacements of the sides of the elements marked in Fig. 1a by "×" and "○", respectively.

Equations (3.1)-(3.4) are Euler's equations for a positive-definite quadratic energy functional. It can be shown that when N, M → ∞ and the linear dimensions of the elements simultaneously approach zero, the solution of system (3.1)-(3.4) converges in the energy norm to the solution of the two-dimensional problem of the theory of elasticity. The procedure that will be described below for the iterative solution of Eqs. (3.1)-(3.4) can be interpreted as the process of successive minimization of the energy functional.

4. Iterative Solution of the Equations of the Sandwich. The sequence $(U^m_{n-1/2})^k$, $(U_n^{m-1/2})^k$, k = 1, 2, which minimizes the energy functional, is constructed as follows. Let the vectors $(U^m_{n-1/2})^k$ be known. Then the vectors $(U_n^{m-1/2})^k$ are found as the solution of Eqs. (3.1), (3.3-3.4). Here, the system decomposes into M independent systems in the quantities $U_n = (U_n^{m-1/2})^k$, n = 1,..., N + 1, m = 2,..., M + 1, each of which can be written in the form

$$\begin{gathered} A_nU_{n-1} + B_nU_n + C_nU_{n+1} = F_n,\ n = 2,\ldots,N, \\ B_1U_1 + C_1U_2 = F_1,\ A_{N+1}U_N + B_{N+1}U_{N+1} = F_{N+1}. \end{gathered} \tag{4.1}$$

Equations (4.1) can be solved by the method of matrix trial run. The resulting vectors $(U_nW^{m-1/2})^k$ and $(U^m_{n-1/2})^k$ satisfy Eqs. (3.1), (3.3)-(3.4) but cannot satisfy Eqs. (3.2). The errors of Eqs. (3.2) $Q_{n-1/2}{}^m = (T_2)_{n-1/2}{}^{m-1/2} + (T_4)_{n-1/2}{}^{m+1/2}$ represent concentrated forces on the sides of the elements at the interfaces. The vectors $(U^m_{n-1/2})^{k+1}$ are found from the equations $Q^m_{n-1/2} = 0$. In these equations, we set $U^m{}_{n-1/2} = (U^m{}_{n-1/2})^{k+1}$, $U_n{}^{m-1/2} = (U_n{}^{m-1/2})^k$.

TABLE 1

Iteration	σ_{22}/σ_0			δ
1	7,6425	4,1991	3,0580	0,2909
2	7,5348	4,1801	3,0124	0,1075
3	7,4221	4,1443	2,9698	0,2469
4	7,3934	4,1288	2,9580	0,1497
5	7,3724	4,1162	2,9489	0,1061
6	7,3436	4,0980	2,9361	0,0817
7	7,3329	4,0913	2,9315	0,0483

In specific numerical calculations, the iterations were continued until the largest error for the region $Q^m_{n-1/2}$, referred to the characteristic stress, became less than a preassigned number ε. We began calculating the solution with the region represented in the form of a single layer consisting of N elements. We then successively doubled the number of layers by dividing each layer by two. When the region was represented as a sandwich of 2M layers, we took as the zeroth approximation for $U^m_{n-1/2}$ the corresponding values calculated for a sandwich of M layers.

4. Examples of Solution of Two-Dimensional Elastic Problems by the Layers Method. The numerical solution of problems with singularities in the stress state is made somewhat difficult by the abrupt reduction in the rate of convergence in the neighborhood of stress concentrations. Below, we present the results of numerical experiments involving the solution of such problems by the iteration method described above.

Tension of a Plane with a Slit. As one example of an iterative solution by the layers method, we examined the determination of the stress–strain state in a plane with a slit (Fig. 2a). The elastic space was cut along the segment $-a \leq x_1 \leq a$, $x_2 = 0$. At infinity, $\sigma_{22} = \sigma$, $\sigma_{11} = \sigma_{12} = 0$. The problem has an analytic solution. In accordance with the latter, the stresses and displacements along the line $x_2 = 0$ are calculated from the formulas

$$\sigma_{22} = 0,\ u = \sigma(1 - \nu)\sqrt{a^2 - x_1^2}/\mu, \quad |x_1| \leq a,$$
$$\sigma_{22} = \sigma x_1/\sqrt{x_1^2 - a^2}, \quad |x_1| > a.$$

The numerical solution was constructed in the rectangular region $\{0 \leq x_1 \leq L,\ 0 \leq x_2 \leq H\}$ (Fig. 2a). Symmetry conditions were prescribed on the sections of the boundary AE and BC: $\sigma_{12} = u_1 = 0$ on AE and $\sigma_{12} = u_2 = 0$ on BC. On AB (the line of the cut), we assumed that the normal and shear stresses were equal to zero. On sections ED and DC, we assigned the stresses and displacements as functions of the point of the boundary corresponding to the analytic solution. We began calculating the solution with representation of the region in the form of a single layer, then successively doubling the number of layers. We performed calculations with different numbers of finite elements in the direction of the x_1 axis and different linear dimensions for the elements in this direction.

Some of the results of the numerical solution are shown in Fig. 2b. These results were obtained with $L/L_0 = 5$, $H/L_0 = 5$, $E/\sigma_0 = 1$, $\varepsilon_0 = \sigma_0/\mu$, $\nu = 0.3$, $\sigma/\sigma_0 = 1$, $\sigma_0 = 1$, $a/L_0 = 1$, $L_0 = 1$. The dimensions of the finite elements in the x_1 direction are shown in Fig. 2a. The smallest elements were located near the edge of the slit and had a linear dimension of 0.04. In the numerical solution, we determined the stresses and displacements averaged over the boundaries of the elements.

Figure 2b shows graphs of the mean (over the element boundaries) normal stresses σ_{22} on a continuation of the cut line. The same figure also shows the mean normal displacements u_2 on this line. The cicles denote values corresponding to the analytic solution. The solid line represents the numerical solution obtained with division of the region into 32 layers. The dashed and dot-dash lines show the solutions for a region divided into 16 and 4 layers, respectively. With a 32-layer sandwich, the mean normal stress on the lower boundary of an element adjacent to the slit differs 2.5% from the exact solution. Regardless of the number of layers into which the region was divided, no more than seven iterations were required to ensure that the largest error in the region $Q^m_{n-1/2}$ did not exceed 0.05.

Table 1 shows values of the normal stresses on the lower boundaries of three adjacent elements next to the slit. Also shown is the maximum (for the region) error δ for a sequence of seven iterations in the case of a 32-layer region. The table gives an idea of the rate of convergence. The results of the solution of the test problem show that the finite elements that were used satisfactorily describe the stress state in the neighborhood of the slit.

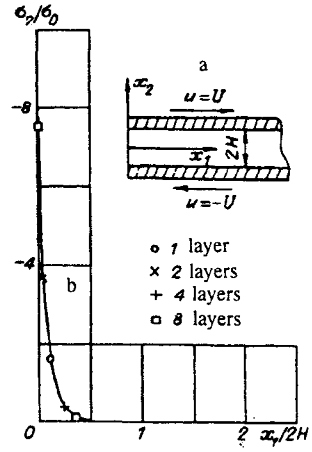

Fig. 5

Problems involving the study of stress states with singularities of the boundary-layer type are of great theoretical and practical interest. Such singularities arise (for example) near the free surface at the interface of layers of materials with different properties. Below, we solve two problems concerning edge effects in the stress state in elastic layers between rigid slabs.

Problem of the Tension of an Elastic Interlayer. A interlayer of thickness 2H is placed in tension by perfectly rigid slabs (Fig. 3a). The normal displacements $u_2 = \pm V$ are assigned on the surfaces of the interface, while the tangential displacement is set equal to zero. A concentration of normal and shear stresses arises near the free surface ($x_1 = 0$) and may cause the interface to separate from the slabs. Figure 3b shows the distribution of the shear stresses along the surface $x_2 =$ H of the interface. Figure 4 shows the normal stresses. The results in these figures were obtained with division of the interface into one, two, four, and eight layers. The distribution of the normal and shear stresses calculated for representation of the interface as a single layer was nearly the same as the distributions calculated for the multiple layers.

Problem of the Shear of an Elastic Interface. An interface of thickness 2H undergoes shear between two perfectly rigid slabs (Fig. 5a). The tangential displacements $u_1 = \pm U$ are assigned on the surfaces of the interface and the normal displacements are taken equal to zero. A concentration of normal and shear stresses develops near the free surface ($x_1 = 0$). Figure 5b shows the distribution of the normal stresses over the surface $x_2 = H$ of the interface when it is divided into different numbers of layers (from one to eight). As in the previous problem, the normal stresses on the surface of the layer can be determined by representing it in the form of a single layer.

These results, as well as the solutions of several other contact problems [2, 3], illustrate the efficiency of the layers method for studying edge effects in elastic interfaces and in other two-dimensional elastic problems with singularities in the stress state.

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